

Determinants of Circulant Matrices with Some Certain Sequences

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Abstract

Let $\{a_k\}$ be a sequence of real numbers defined by an m th order linear homogeneous recurrence relation. In this paper we obtain a determinant formula for the circulant matrix $A = \text{circ}(a_1, a_2, \dots, a_n)$, providing a generalization of determinantal results in papers of Bozkurt [2], Bozkurt and Tam [3], and Shen, et al. [8].

Keywords: circulant matrix, determinant, Fibonacci sequence, Lucas sequence, tribonacci sequence

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1. Introduction

The circulant matrix $V = \text{circ}(v_1, v_2, \dots, v_n)$ associated to real numbers v_1, v_2, \dots, v_n is the $n \times n$ matrix

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_n & v_1 & \cdots & v_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_2 & v_3 & \cdots & v_1 \end{pmatrix}.$$

Circulant matrices are one of the most interesting members of matrices. They have elegant algebraic properties. For example, $\text{Circ}(n)$ is an algebra on \mathbb{C} . Let ϵ be a primitive n^{th} root of unity. For each $0 \leq k \leq n-1$, $\lambda_k = \sum_{j=0}^{n-1} v_j \epsilon^{kj}$ is an eigenvalue of $V = \text{circ}(v_1, v_2, \dots, v_n)$ and the corresponding eigenvector is $x_k = \frac{1}{\sqrt{n}}(1, \epsilon^k, \epsilon^{2k}, \dots, \epsilon^{(n-1)k}) \in \mathbb{C}^n$. Indeed, all circulant matrices have the same ordered set of orthonormal eigenvectors $\{x_k\}$. Besides, $\det V = \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} v_j \epsilon^{kj} \right)$. The reader can consult the text of Davis [4] for further properties of circulant matrices. On the other hand, circulant matrices have a widespread applications in many parts of mathematics. The excellent survey

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paper [6] includes many applications of circulant matrices in various areas of mathematics. Also, they have applications in signal processing, the study of cyclic codes for error corrections [5] and in quantum mechanics [1].

Recently, many authors have investigated some properties of circulant matrices associated to so famous integer sequences, for example, the Fibonacci sequence and the Lucas sequence. Let $a, b, p, q \in \mathbb{Z}$. Define a sequence (U_n) by the second order recurrence relation

$$U_n = pU_{n-1} + qU_{n-2} \quad (1.1)$$

($n \geq 3$) with initial conditions $U_1 = a$ and $U_2 = b$. Taking $(p, q, a, b) = (1, 1, 1, 1)$, $(1, 1, 1, 3)$, $(1, 2, 1, 1)$ and $(1, 2, 1, 3)$, (U_n) becomes the Fibonacci sequence (F_n) , the Lucas sequence (L_n) , the Jacobsthal sequence (J_n) and the Jacobsthal-Lucas sequence (j_n) , respectively. In 1970 Lind [7] obtained a formula for the determinant of $F = \text{circ}(F_r, F_{r+1}, \dots, F_{r+n-1})$ ($r \geq 1$). In 2005 Solak [9] investigated matrix norms of $F = \text{circ}(F_1, F_2, \dots, F_n)$ and $L = \text{circ}(L_1, L_2, \dots, L_n)$. In 2011 Shen, Cen and Hao [8] showed that

$$\det(F) = (1 - F_{n+1})^{n-1} + F_n^{n-2} \sum_{k=1}^{n-1} F_k \left(\frac{1 - F_{n+1}}{F_n} \right)^{k-1}$$

and

$$\det(L) = (1 - L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum_{k=1}^{n-1} (L_{k+2} - 3L_{k+1}) \left(\frac{1 - L_{n+1}}{L_n - 2} \right)^{k-1}$$

Recently, Bozkurt and Tam [3] have obtained determinant formulae for $J = \text{circ}(J_1, J_2, \dots, J_n)$ and $\mathbb{J} = \text{circ}(j_1, j_2, \dots, j_n)$ using the same method. Then Bozkurt [2] has given a generalization of these determinant formulae as

$$\begin{aligned} \det(U) &= (a^2 - bU_n)(a - U_{n+1})^{n-2} \\ &\quad + \sum_{k=2}^{n-1} (aU_{k+1} - bU_k)(a - U_{n+1})^{k-2} (qU_n - b + qa)^{n-k}, \end{aligned} \quad (1.2)$$

where $\{U_k\}$ is the sequence in (1.1).

In all of the above-mentioned papers authors calculated determinants of circulant matrices associated to a sequence defined by a second order recurrence relation by using the same method. In this paper we generalize determinantal results of these papers for certain sequences defined by a recurrence relation of order $m \geq 1$.

2. The Main Result

Let c_1, c_2, \dots, c_m be real numbers and $c_m \neq 0$. Consider the sequence $\{a_k\}$ defined by the m th order linear homogenous recurrence relation

$$a_k = c_1 a_{k-1} + c_2 a_{k-1} + \cdots + c_m a_{k-m} \quad (k \geq m+1) \quad (2.1)$$

with initial conditions

$$a_1, a_2, \dots, a_m, \quad (2.2)$$

which are given real numbers. Let $n > m$ and $A = \text{circ}(a_1, a_2, \dots, a_n)$. Let A_{ij} be the ij -entry of A . It is clear that $A_{ij} = a_{j-i+1}$ if $j \geq i$ and $a_{n+j-i+1}$ otherwise. On the other hand, for simplicity, we write $A_{ij} = a_{(j-i+1)}$ in both case. Our main goal is to reduce the order n of the determinant of A and to calculate it in a simpler way. In order to perform this, first we define an $n \times n$ matrix $P = (P_{ij})$, where

$$P_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i + j = n + 2, \\ -c_m & \text{if } i = m + 1 \text{ and } j = 1, \\ -c_t & \text{if } i + j - t = n + 2 \text{ and } i \geq m + 1 \text{ and } 1 \leq t \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

Then the ij -entry of the product of P and A is

$$(PA)_{ij} = \begin{cases} A_{1j} & \text{if } i = 1, \\ A_{n-i+2,j} & \text{if } 2 \leq i \leq m, \\ \alpha_t & \text{if } i + j = n + t + 1 \text{ and } 1 \leq t \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \alpha_t &= A_{n-m+1, n-m+t} - c_1 A_{n-m+2, n-m+t} \\ &\quad - \cdots - c_{m-1} A_{n, n-m+t} - c_m A_{1, n-m+t}. \end{aligned} \quad (2.3)$$

Now, we define a sequence $\{b_s^{(r)}\}$ for every $r = 1, 2, \dots, m-1$ by the recurrence relation

$$b_s^{(r)} = -\frac{\alpha_2}{\alpha_1} b_{s-1}^{(r)} - \frac{\alpha_3}{\alpha_1} b_{s-2}^{(r)} - \cdots - \frac{\alpha_m}{\alpha_1} b_{s-m+1}^{(r)} \quad (s \geq m) \quad (2.4)$$

with initial conditions

$$b_i^{(r)} = \delta_{i,r}, \quad (2.5)$$

the Kronecker delta, for $i = 1, 2, \dots, m-1$. We form another $n \times n$ matrix $Q = (Q_{ij})$ such that

$$Q_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i + j = n + 2, \\ b_{n-i+1}^{(j-1)} & \text{if } 2 \leq i \leq n - m + 1 \text{ and } 2 \leq j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$(PAQ)_{ij} = \begin{cases} A_{1,1} & \text{if } i = j = 1, \\ A_{n-i+2,1} & \text{if } 2 \leq i \leq m \text{ and } j = 1, \\ \sum_{k=2}^n A_{1k} b_{n-k+1}^{(j-1)} & \text{if } i = 1 \text{ and } 2 \leq j \leq m, \\ \sum_{k=2}^n A_{n-i+2,k} b_{n-k+1}^{(j-1)} & 2 \leq i, j \leq m, \\ \alpha_k & \text{if } i, j > m \text{ and } 1 \leq k \leq m \text{ and } i - j = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $A_{ij} = a_{j-i+1}$ if $j \geq i$ and $a_{n+j-i+1}$ otherwise and that we write $A_{i,j} = a_{(j-i+1)}$ for simplicity. Also, it is clear that $\det P = \det Q = (-1)^{\frac{n(n+1)}{2}-1}$ and $\alpha_1 = a_1 - a_{n+1}$. Finally, we get the following lemma.

Lemma 2.1. *Let $\{a_k\}$ be the sequence defined by the recurrence relation in (2.1) with initial conditions in (2.2), $n > m$ and $A = \text{circ}(a_1, a_2, \dots, a_n)$. Then*

$$\det(A) = (a_1 - a_{n+1})^{n-m} \sum_{k_1=2}^n \cdots \sum_{k_{m-1}=2}^n \begin{vmatrix} a_1 & a_{(k_1)} & \cdots & a_{(k_{m-1})} \\ a_2 & a_{(k_1+1)} & \cdots & a_{(k_{m-1}+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{(k_1+m-1)} & \cdots & a_{(k_{m-1}+m-1)} \end{vmatrix} \prod_{i=1}^{m-1} b_{n-k_i+1}^{(i)}, \quad (2.6)$$

where sequences $\{b_s^{(r)}\}$ are defined by the recurrence relation in (2.4) with initial conditions in (2.5).

Indeed, the determinant formula for $A = \text{circ}(a_1, a_2, \dots, a_n)$ in Lemma 2.1 is not effective but we obtain it by generalizing the common method of papers [8, 3, 2] for the sequence $\{a_k\}$ defined by a recurrence relation of order $m \geq 1$. To illustrate our goal we consider the well-known tribonacci sequence. The tribonacci sequence $\{a_k\}$ is defined by the recurrence relation

$$a_k = a_{k-1} + a_{k-2} + a_{k-3} \quad (k \geq 4)$$

with initial conditions $a_1 = 1, a_2 = 1, a_3 = 2$. For convenience, we take $a_0 = 0$.

Corollary 2.2. *Let $\{a_k\}$ be the tribonacci sequence, $n > 3$ and $A = \text{circ}(a_1, a_2, \dots, a_n)$. Then*

$$\begin{aligned} \det(A) = & (1 - a_{n+1})^{n-3} \left(\sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} (a_{i-2}a_{j-1} - a_{i-1}a_{j-2}) \left(\frac{\alpha_3}{\alpha_1}\right)^{n-j-1} b_{j-i+2}^{(1)} \right. \\ & + \sum_{i=2}^{n-2} ((a_{i-2} + a_{i-1}) + a_{n-1}(a_{i+2} - 2a_{i+1}) + a_n(2a_i - a_{i+2})) b_{n-i+1}^{(1)} \\ & \left. + \sum_{i=2}^{n-2} (-a_{i-1} + a_n(a_{i+2} - 2a_{i+1})) \frac{\alpha_1}{\alpha_3} b_{n-i+2}^{(1)} + (2a_n^2 - 2a_n - a_{n-1} + 1) \right). \end{aligned}$$

Proof. Let $\{a_k\}$ in Lemma 2.1 be the tribonacci sequence. Then clearly $m = 3$, $a_1 = a_2 = 1, a_3 = 2, \alpha_1 = 1 - a_{n+1}$ and by Lemma 2.1, we have

$$\det(A) = (1 - a_{n+1})^{n-3} \sum_{i=2}^n \sum_{j=2}^n \begin{vmatrix} 1 & a_{(i)} & a_{(j)} \\ 1 & a_{(i+1)} & a_{(j+1)} \\ 2 & a_{(i+2)} & a_{(j+2)} \end{vmatrix} b_{n-i+1}^{(1)} b_{n-j+1}^{(2)}.$$

We denote the 3×3 determinant in the summation by $\Delta((i), (j))$. It is clear that $\Delta((i), (i)) = 0$ and $\Delta((j), (i)) = -\Delta((i), (j))$. Also, we have $\Delta((i), (j)) = \Delta(i, j)$ if $1 \leq i, j \leq n-3$. Thus

$$\det(A) = (1 - a_{n+1})^{n-3} \sum_{i=2}^{n-1} \sum_{j=i+1}^n \Delta((i), (j)) (b_{n-i+1}^{(1)} b_{n-j+1}^{(2)} - b_{n-j+1}^{(1)} b_{n-i+1}^{(2)}).$$

Now, sequences $\{b_k^{(1)}\}$ and $\{b_k^{(2)}\}$ are generated by the recurrence relation in (2.4) with different initial conditions, all of which are given in (2.5). The characteristic equation of the recurrence relation in (2.4) is $\alpha_1 r^2 + \alpha_2 r + \alpha_3 = 0$, where $\alpha_1 = 1 - a_{n+1}$, $\alpha_2 = -a_n - a_{n-1}$ and $\alpha_3 = -a_n$. Since $\alpha_2^2 - 4\alpha_1\alpha_3 < (-a_n + a_{n-1})(3a_n + a_{n-1}) < 0$ for all $n \geq 1$, the characteristic equation has two distinct complex roots, say λ and μ . Finally, Binet's formulae for sequences $b_k^{(1)}$ and $b_k^{(2)}$ are $b_k^{(1)} = \frac{\lambda\mu}{\mu-\lambda}(\lambda^{k-2} - \mu^{k-2})$ and $b_k^{(2)} = \frac{1}{\lambda-\mu}(\lambda^{k-1} - \mu^{k-1})$, respectively. Using Binet's formulae we have the identity

$$b_k^{(1)}b_t^{(2)} - b_t^{(1)}b_k^{(2)} = \left(\frac{\alpha_3}{\alpha_1}\right)^{t-2} b_{k-t+2}^{(1)},$$

where $k \geq t$. Thus, we have

$$\begin{aligned} \det(A) &= (1 - a_{n+1})^{n-3} \left(\sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} \Delta(i, j) \left(\frac{\alpha_3}{\alpha_1}\right)^{n-j-1} b_{j-i+2}^{(1)} \right. \\ &\quad + \sum_{i=2}^{n-2} \Delta(i, (n-1)) b_{n-i+1}^{(1)} + \sum_{i=2}^{n-2} \Delta(i, (n)) \frac{\alpha_1}{\alpha_3} b_{n-i+2}^{(1)} \\ &\quad \left. + \Delta((n-1), (n)) \frac{\alpha_1}{\alpha_3} b_3^{(1)} \right). \end{aligned}$$

The proof follows from equalities

$$\begin{aligned} \Delta(i, j) &= a_{i-2}a_{j-1} - a_{i-1}a_{j-2}, \\ \Delta(i, (n-1)) &= (2a_n - 1)a_i + (1 - 2a_{n-1})a_{i+1} + (a_{n-1} - a_n)a_{i+2}, \\ \Delta(i, (n)) &= a_i + (1 - 2a_n)a_{i+1} + a_{n-1}a_{i+2}, \\ \Delta((n-1), (n)) \frac{\alpha_1}{\alpha_3} b_3^{(1)} &= 2a_n^2 - 2a_n - a_{n-1} + 1. \end{aligned}$$

□

We cannot state that the determinant formula in Corollary 2.2 is elegant but it reduces an $n \times n$ determinant to a double sum.

Corollary 2.3 ([2], Theorem 1). *Let $\{U_k\}$ be the sequence defined by the recurrence relation given in (1.1) with initial conditions $U_1 = a, U_2 = b, n > 3$ and $A = \text{circ}(U_1, U_2, \dots, U_n)$. Then*

$$\det(U) = (a^2 - bU_n)(a - U_{n+1})^{n-2} + \sum_{k=2}^{n-1} (aU_{k+1} - bU_k)(a - U_{n+1})^{k-2} (qU_n - b + pa)^{n-k}.$$

Proof. Let $\{a_k\}$ in Lemma 2.1 be the sequence $\{U_k\}$ given in (1.1) with initial conditions $U_1 = a$ and $U_2 = b$. Then $\alpha_1 = a - U_{n+1}$, $\alpha_2 = b - pU_1 - qU_n$ and hence $b_i^{(1)} = (-\alpha_2/\alpha_1)^{i-1}$. Thus, by Lemma 2.1, we have

$$\begin{aligned} \det(A) &= (a - U_{n+1})^{n-2} \sum_{k=2}^n \begin{vmatrix} a & U_{(k)} \\ b & U_{(k+1)} \end{vmatrix} b_{n-k+1}^{(1)} \\ &= (a - U_{n+1})^{n-2} [(a^2 - U_n b) + \sum_{k=2}^{n-1} \begin{vmatrix} a & U_k \\ b & U_{k+1} \end{vmatrix} b_{n-k+1}^{(1)}] \\ &= (a - U_{n+1})^{n-2} [(a^2 - bU_n) + \sum_{k=2}^{n-1} (aU_{k+1} - bU_k) (-\frac{qU_n - b + pa}{a - U_{n+1}})^{n-k}]. \end{aligned}$$

A simple calculation completes the proof. \square

Renaming terms of sequence $\{U_k\}$ as $\{W_{k-1}\}$ we obtain the same formula in Theorem 1 of Bozkurt's paper [2]. Also, by choosing convenient values for p, q, a and b in Corollary 2.3 we can obtain all determinant formulae in [3, 8]. Taking $(p, q, a, b) = (1, 1, 1, 1), (1, 1, 1, 3), (1, 2, 1, 1)$ and $(1, 2, 1, 3)$, we have Theorems 2.1 and 3.1 of [8] and Theorems 2.1 and 2.2 of [3], respectively. Also, by Lemma 2.1, we can easily evaluate the determinant of $A = \text{circ}(a, a^2, a^3, \dots, a^n)$, where a is a nonzero real number, as $\det(A) = a^n(1 - a^n)^{n-1}$.

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